2.4.3 Volume using Integral

▶ Volume of a Cylindrical Solid: A cylindrical solid is <u>bounded above</u> by a surface $z = f(x, y)$ and <u>bounded below</u> by a plane region R on $xy-plane$ and the <u>sides bounded</u> by straight lines parallel to $z-axis$. Then volume of the solid is

$$\boxed{V = \int\int_E \int_{z=0}^{f(x,y)} dxdydz = \int\int_E f(x, y)\, dxdy}$$

▶ Volume Enclosed by two Surfaces: If a solid is <u>bounded above</u> by a surface $z = f_2(x, y)$ and <u>bounded below</u> by the surface $z = f_1(x, y)$ and both the surfaces are above the $xy-plane$. Also the <u>sides bounded</u> by straight lines parallel to $z-axis$. Then volume of the solid is

$$\boxed{V = \int\int_E [f_2(x, y) - f_1(x, y)]\, dxdy}$$

Here E is the projection of both the surfaces on $xy-plane$. Also here both f_1, f_2 must be positive, continuous and $f_2 \geq f_1$ on E.

▶ Volume Enclosed by a Closed Surface: Let S be a closed surface and any straight line parallel to $z-axis$ cut it in almost two points. The surface S have two parts $z = \phi_2(x, y)$ upper part and $z = \phi_1(x, y)$ is the lower part. Then volume of the solid is

$$\boxed{V = \int\int_E [\phi_2(x, y) - \phi_1(x, y)]\, dxdy}$$

Here E is the projection of S on $xy-plane$.

Example 2.18. *Evaluate* $\int\int\int_V \frac{dxdydz}{(x+y+z+1)^3}$ *where* V *is the tetrahedron bounded by the planes* $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$.

$$\Rightarrow \int\int\int_V \frac{dxdydz}{(x+y+z+1)^3} = \int_{x=0}^{1} dx \int_{y=0}^{1-x} dy \int_{z=0}^{1-x-y} \frac{1}{(x+y+z+1)^3} dz =$$
$$\frac{1}{2}\int_{x=0}^{1} dx \int_{y=0}^{1-x} dy \Big[\frac{1}{(1+x+y)^2} - \frac{1}{4}\Big] = \frac{1}{2}\int_{x=0}^{1} dx \int_{y=0}^{1-x} \Big[\frac{1}{(1+x+y)^2} - \frac{1}{4}\Big] dy =$$
$$\frac{1}{2}\int_{x=0}^{1} \Big[\frac{1}{1+x} - \frac{3}{4} + \frac{x}{4}\Big] dx = \frac{1}{2}\Big[\ln 2 - \frac{5}{8}\Big].$$

[Do It Yourself] 2.100. *Find the volume of the region in the first octant* $(x, y, z \geq 0)$ *bounded by the cylinder* $x^2 + y^2 = 4$ *and the planes* $z = 2,\ y + z = 4$.

$$[Hint:\ V = \int_{z=2}^{4-y} dz \int_{x=0}^{2} dx \int_{y=0}^{\sqrt{4-x^2}} dy = \int_{x=0}^{2} dx \int_{y=0}^{\sqrt{4-x^2}} (2-y)dy = 2\pi - \frac{8}{3}].$$

[Do It Yourself] 2.102. *Consider the region S enclosed by the surface $z = y^2$ and the planes $z = 1$, $x = 0$, $x = 1$, $y = -1$ and $y = 1$. The volume of S is*
(A) $1/3$ *(B)* $2/3$ *(C)* 1 *(D)* $4/3$.

$$[Hint: \ Using\ Rule\ 2: \ V = \int_{x=0}^{1} dx \int_{y=-1}^{1} (1-y^2)dy = \int_{y=-1}^{1} (1-y^2)dy = 4/3].$$

[Do It Yourself] 2.103. *Show that the volume common to the cylinder $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ is $16a^3/3$.*

$$[Hint: \ Use\ Rule\ 1: \ V = 2\int_{x=-a}^{a} dx \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}dy = 4\int_{x=-a}^{a} (a^2-x^2)dx].$$

2.5 Beta and Gamma Functions

- **Gamma function**: $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx \ (n > 0)$, provided the integral converges.
- $\Gamma(n+1) = n\Gamma(n), \ n > 0.$
- $\Gamma(m) = (m-1)!, \ for\ positive\ integer\ m..$
- $\Gamma(1) = 1, \ \Gamma(\frac{1}{2}) = \sqrt{\pi}.$
- $\Gamma(p)\Gamma(1-p) = \pi\csc(p\pi), \ 0 < p < 1.$
- $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \Gamma(\frac{1}{3})\Gamma(1-\frac{1}{3}) = \pi\csc(\frac{\pi}{3}) = \frac{2\pi}{\sqrt{3}}.$
- **Beta function - Form 1**: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \ (m,n > 0)$, provided the integral converges.
- **Beta function - Form 2**: $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx \ (m,n > 0)$, provided the integral converges.
- **Beta function - Form 3**: $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}x\cos^{2n-1}x dx \ (m,n > 0)$, provided the integral converges.
- **Beta and Gamma Relation**: $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$

[Do It Yourself] 2.111. *Let $J = \frac{1}{\pi}\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{3}{2}}dt$. Then what is the value of J?*

[Do It Yourself] 2.112. *Show the following results*

1. $\int_0^\infty e^{-ax}x^{n-1}dx = \frac{\Gamma(n)}{a^n}, \ n, a > 0.$

2. $\Gamma(n+1) = n\Gamma(n)$.

3. $\Gamma(1) = 1,\ \Gamma(1/2) = \sqrt{\pi}$.

4. $\beta(m,n) = \beta(n,m)$.

5. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x dx\ (m,n > 0)$.

[Do It Yourself] 2.113. *Using the transformation* $x = \frac{y}{1+y}$ *show that*
$\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx\ (m,n > 0)$.

[Do It Yourself] 2.114. *Show that* $\int_0^{\pi/2} \sin^p x \cos^q x dx = \frac{1}{2}\beta(\frac{p+1}{2}, \frac{q+1}{2})$.

[Do It Yourself] 2.115. *Find* $\int_0^{\infty} e^{-x^2} dx,\ \int_0^{\infty} x^m e^{-x^n} dx,\ \int_0^{\infty} \sqrt{x} e^{-x^3} dx$.

[Do It Yourself] 2.116. *Find* $\int_0^1 x^3(1-x^7)^2 dx,\ \int_0^1 \sqrt{1-x^4} dx$.

[Do It Yourself] 2.117. *Find* $\int_0^{\pi/2} \sin^5 x dx,\ \int_0^{\pi/2} \sin^6 x dx,\ \int_0^{\pi/2} \cos^7 x dx,\ \int_0^{\pi/2} \cos^3 x dx$.